

THE EXTENDED TODA HIERARCHY

GUIDO CARLET, BORIS DUBROVIN, AND YOUJIN ZHANG

ABSTRACT. Using construction of logarithm of a difference operator, we present the Lax pair formalism for certain extension of the continuous version of the classical Toda lattice hierarchy, provide a well defined notion of tau function for its solutions, and give an explicit formulation of the relationship between the $\mathbb{C}P^1$ topological sigma model and the extended Toda hierarchy. We also establish an equivalence of the latter with certain extension of the nonlinear Schrödinger hierarchy.

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1. INTRODUCTION

The Toda lattice equation [28]

$$\ddot{q}_n = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}}, \quad -\infty < n < \infty, \quad (1.1)$$

is one of the prototypical integrable systems that plays significant role in classical and quantum field theory. The Toda lattice hierarchy consists of infinitely many evolutionary differential-difference equations commuting with (1.1). In this paper we study this hierarchy from the point of view of 2D topological field theory. One of the first lessons of this approach [4], [31] is that, one is to replace the discrete variable n by a continuous one. The result of such “interpolation” is the following equation for the function $q = q(x, t)$

$$\epsilon^2 q_{tt} = e^{q(x-\epsilon)-q(x)} - e^{q(x)-q(x+\epsilon)}. \quad (1.2)$$

In this equation the cosmological constant plays the role of the independent variable x , the formal small parameter ϵ is called the string coupling constant. Similar interpolation can be applied to the whole Toda lattice hierarchy. It is conjectured that the partition function of the $\mathbb{C}P^1$ topological sigma model as the function of the coupling constants of the theory is the tau function of a particular solution of certain extension of the interpolated Toda lattice hierarchy. Under such identification the

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coupling constant corresponding to the identity primary field $\phi_1 \in H^0(\mathbb{C}P^1)$ serves as the spatial variable and that of the remaining primary $\phi_2 \in H^2(\mathbb{C}P^1)$ and of the gravitational descendent fields correspond to the time variables of the hierarchy. Such an extension of the Toda lattice hierarchy is formulated independently in [16], [33], and the above conjecture is known to be true up to genus one approximation [5], [9], [12]–[14], [33]. The extended Toda lattice hierarchy is formulated in [16], [33] by using the bihamiltonian structure of the original Toda lattice hierarchy, and is defined by the bihamiltonian recursion relation. We call this hierarchy *the extended Toda hierarchy* in this paper.

To an expert in the theory of integrable systems that might be less motivated by the eventual applications of the Toda hierarchy to the theory of Gromov–Witten invariants, the importance of considering the extended Toda hierarchy can also be explained by means of the following argument. The flows of the usual Toda hierarchy form a complete family, i. e., they span the space of vector fields commuting with (1.1). This fails to be true for the interpolated Toda lattice hierarchy. Indeed, already the spatial translations $x \mapsto x + c$ do not belong to the linear span of the Toda lattice flows. One can show, using the technique of [7] that the flows of extended Toda lattice hierarchy form a complete family of flows commuting with (1.2).

Two important aspects of the theory of extended Toda hierarchy remained unclear, after [16], [33]. The missing points were the Lax pair formalism and a well defined notion of tau function for an arbitrary solution of the extended hierarchy. A Lax pair formalism is crucial both for the theory of integrability of the hierarchy and for its applications in physics, while a well defined notion of tau function for solutions of the hierarchy is needed to formulate explicitly the relation between the extended Toda hierarchy and the $\mathbb{C}P^1$ topological sigma model, or equivalently, to the theory of Gromov–Witten invariants of $\mathbb{C}P^1$ and their gravitational descendants.

We present in Section 2 a Lax pair formalism of the extended Toda hierarchy. By using this Lax pair formalism we are able to express in Section 3 the densities of the Hamiltonians of the hierarchy in terms of the Lax operator, and in Section 4 we give the definition of the tau function for solutions of the hierarchy by using the result of Section 3. In Section 5 we show that the extended Toda hierarchy is equivalent to certain extension of the nonlinear Schrödinger hierarchy. In the last Section we discuss the relation of the $\mathbb{C}P^1$ topological sigma model with the extended Toda hierarchy.

2. LOGARITHM OF A DIFFERENCE OPERATOR AND FORMULATION OF THE EXTENDED TODA HIERARCHY

The Toda lattice equation (1.1) describes the motion of one-dimensional particles with exponential interaction of neighbors [28]. A crucial aspect among the integrability properties of this equation is its Lax pair formalism given by H. Flaschka [15] and S. Manakov [24]. By introducing the new dependent variables

$$v_n = -\frac{\partial q_n}{\partial t}, \quad u_n = q_{n-1} - q_n, \quad (2.1)$$

we can rewrite the Toda lattice equation in the form

$$\frac{\partial v_n}{\partial t} = e^{u_{n+1}} - e^{u_n}, \quad \frac{\partial u_n}{\partial t} = v_n - v_{n-1}, \quad n \in \mathbb{Z}. \tag{2.2}$$

Let Λ be the shift operator defined by

$$\Lambda f_n = f_{n+1}$$

for any function f on the one dimensional infinite lattice. The Lax operator¹ is defined by

$$\bar{L} = \Lambda + v_n + e^{u_n} \Lambda^{-1} \tag{2.3}$$

and the Toda lattice equation can be recast into the form

$$\frac{\partial \bar{L}}{\partial t} = [\Lambda + v_n, \bar{L}]. \tag{2.4}$$

Here the square bracket stands for the usual commutator of two operators. Related to the Toda lattice equation there is an infinite family of mutually commuting flows of the form

$$\frac{\partial \bar{L}}{\partial t_p} = \frac{1}{(p+1)!} [(\bar{L}^{p+1})_+, \bar{L}], \quad p \geq 0, \quad \frac{\partial}{\partial t_q} \frac{\partial \bar{L}}{\partial t_p} = \frac{\partial}{\partial t_p} \frac{\partial \bar{L}}{\partial t_q}. \tag{2.5}$$

This family of evolutionary differential-difference equations is the so-called Toda lattice hierarchy. Clearly for $p = 0$ the equation (2.5) coincides with (2.4).

We are to define certain extension of the Toda lattice by constructing another infinite family of evolutionary equations that commute with each other and with the flows of the original Toda lattice hierarchy. To this end, we first replace the discrete variable n by a continuous variable x . By interpolating we introduce the dependent variables $u(x), v(x)$ such that

$$u_n = u(\epsilon n), \quad v_n = v(\epsilon n). \tag{2.6}$$

Here ϵ is a formal parameter that can be viewed as the lattice mesh. We will also use an alternative notation for the dependent variables

$$w^1 := v, \quad w^2 := u \tag{2.7}$$

and we will denote $w = (w^1, w^2)$ the two-component vector. Then the Toda lattice hierarchy for the functions $w^\alpha(x, t_0, t_1, \dots)$, $\alpha = 1, 2$, can be recast into the form

$$\epsilon \frac{\partial L}{\partial t^{2,p}} = \frac{1}{(p+1)!} [(L^{p+1})_+, L], \quad p \geq 0. \tag{2.8}$$

Here the Lax operator L acting on smooth functions on the line is defined by

$$L = \Lambda + v(x) + e^{u(x)} \Lambda^{-1} \tag{2.9}$$

with Λ being defined now as the shift operator

$$\Lambda = e^{\epsilon \partial_x}$$

and the time variables $t^{2,p}$ are obtained from t_p by rescaling $t^{2,p} = \epsilon t_p$. We call this hierarchy the *Toda hierarchy*.

¹We use here not the original Lax operator introduced in [15], [24] but the one obtained from [15], [24] by a gauge transformation.

Let us denote \mathcal{R} the ring of formal power series of the form $\sum_{k \geq 0} f_k \epsilon^k$, where f_k are polynomials of the variables $v(x)$, $u(x)$, $e^{\pm u(x)}$ and the x -derivatives of v , u . The gradation on \mathcal{R} is defined by

$$\deg v^{(m)} = 1 - m, \quad \deg u^{(m)} = -m, \quad \deg e^u = 2, \quad \deg \epsilon = 1, \quad m \geq 0. \quad (2.10)$$

Here

$$v^{(m)} = \partial_x^m v, \quad u^{(m)} = \partial_x^m u. \quad (2.11)$$

The equations of Toda hierarchy will be considered as \mathcal{R} -valued vector fields. For example, the interpolated Toda lattice equation has the form

$$\begin{aligned} \frac{\partial v}{\partial t^{2,0}} &= \frac{1}{\epsilon} (e^{u(x+\epsilon)} - e^{u(x)}) = \sum_{k \geq 0} \frac{\epsilon^k}{(k+1)!} \partial_x^{k+1} e^u \\ \frac{\partial u}{\partial t^{2,0}} &= \frac{1}{\epsilon} (v(x) - v(x-\epsilon)) = \sum_{k \geq 0} (-1)^{k+1} \frac{\epsilon^k}{(k+1)!} \partial_x^{k+1} v. \end{aligned} \quad (2.12)$$

Following [7], we will treat equations of this class as infinite order evolutionary PDEs. For the sake of brevity they will also be called PDEs in subsequent considerations. The solutions of such PDEs will be considered in the class of formal power series in ϵ .

The dressing operators P and Q (see [29])

$$P = \sum_{k \geq 0} p_k \Lambda^{-k}, \quad Q = \sum_{k \geq 0} q_k \Lambda^k, \quad p_0 = 1, \quad (2.13)$$

can be formally defined by the following identities in the ring of Laurent series in Λ^{-1} and Λ respectively:

$$L = P \Lambda P^{-1} = Q \Lambda^{-1} Q^{-1}. \quad (2.14)$$

Note that the coefficients p_k and q_k of the dressing operators do not belong to the ring \mathcal{R} but to a certain extension of it (see [29]). The dressing operators are defined up to the multiplication from the right by operators of the form $1 + \sum_{k \geq 1} \hat{p}_k \Lambda^{-k}$ and $\sum_{k \geq 0} \hat{q}_k \Lambda^k$ respectively, where \hat{p}_k , \hat{q}_k are some constants.

To construct an extension of the Toda hierarchy we need to introduce the following notion of the logarithm of the Lax operator L :

$$\log L := \frac{1}{2} (P \epsilon \partial_x P^{-1} - Q \epsilon \partial_x Q^{-1}). \quad (2.15)$$

Remarkably the above ambiguity in the choice of dressing operators is cancelled in the definition of the operator $\log L$. Moreover, the coefficients of the operator $\log L$ do belong to \mathcal{R} as the following theorem guarantees:

Theorem 2.1. *The operator $\log L$ has the following expression*

$$\log L = \sum_{k \in \mathbb{Z}} g_k \Lambda^k, \quad g_k = g_k(w, w_x, w_{xx}, \dots; \epsilon) \in \mathcal{R}. \quad (2.16)$$

Proof. Let us first consider the operator $\epsilon P_x P^{-1}$ where $P_x = \sum_{k \geq 1} p_{k,x} \Lambda^{-k}$. It has the expression

$$\epsilon P_x P^{-1} = \sum_{k \geq 1} a_k \Lambda^{-k}. \tag{2.17}$$

From the definition of the dressing operator P it follows that

$$[\epsilon P_x P^{-1}, L^m] = \epsilon \partial_x L^m, \quad m \geq 1. \tag{2.18}$$

For any operator $A = \sum Y_k \Lambda^k$ let us define its residue by

$$\text{res } A = Y_0. \tag{2.19}$$

By taking residue on both sides of (2.18) with $m = 1$ we get

$$a_1(x + \epsilon) - a_1(x) = -\epsilon \partial_x v(x), \tag{2.20}$$

which shows that $a_1 \in \mathcal{R}$ (see the explicit formula (2.23) below). By induction on the index of a_k and by taking residue on both sides of (2.18) for general m we show that $a_k \in \mathcal{R}$ for $k \geq 1$.

To finish the proof of the theorem, we need to obtain similar result for the operator $Q_x Q^{-1}$. From (2.13), (2.14) we know that the function q_0 that appears in the expression of Q satisfies the relations

$$\frac{q_0(x)}{q_0(x - \epsilon)} = e^{u(x)}, \quad \frac{q_{0,x}(x)}{q_0(x)} - \frac{q_{0,x}(x - \epsilon)}{q_0(x - \epsilon)} = u_x, \tag{2.21}$$

from which it follows that the coefficients of the operator $\tilde{L} = q_0^{-1} L q_0$ belong to \mathcal{R} . Denote $\tilde{Q} = q_0^{-1} Q$. We have by definition the relation

$$\tilde{L} = \tilde{Q} \Lambda^{-1} \tilde{Q}^{-1}. \tag{2.22}$$

By using the identity (2.22) we can show, as we did for the operator $\epsilon P_x P^{-1}$ above, that the operator $\epsilon \tilde{Q}_x \tilde{Q}^{-1}$ has the expression

$$\epsilon \tilde{Q}_x \tilde{Q}^{-1} = \sum_{k \geq 1} b_k \Lambda^k, \quad b_k \in \mathcal{R}.$$

Then the theorem follows from (2.21) and the identities

$$\log L = \frac{1}{2} (\epsilon Q_x Q^{-1} - \epsilon P_x P^{-1}), \quad Q = q_0 \tilde{Q}.$$

The theorem is proved. □

The proof of the above theorem also gives an algorithm of computing the coefficients g_k of the operator $\log L$ expanded in the form (2.16). Indeed, from (2.20) we obtain

$$a_1(x) = - \sum_{k \geq 0} \frac{B_k}{k!} (\epsilon \partial_x)^k v(x) + c \tag{2.23}$$

where the coefficients B_k are the Bernoulli numbers and c is an integration constant. If we set $v = e^u = 0$ in the Lax operator L , then the coefficients of the dressing operator P must be constants, and in this situation $P_x P^{-1} = 0$. This fact implies that the integration constant c must be equal to zero. Now, if we already obtained

the expression for the first $n - 1$ coefficients of $\epsilon P_x P^{-1}$ that is expanded in the form (2.17), then the identity

$$\text{res}([\epsilon P_x P^{-1}, L^n] - \epsilon \partial_x L^n) = 0$$

can be written in the form

$$a_n(x + n\epsilon) - a_n(x) = \epsilon \partial_x W$$

for some $W \in \mathcal{R}$. So we have

$$a_n(x) = \sum_{k \geq 0} \frac{B_k}{k!} (n\epsilon \partial_x)^k W.$$

Here the integration constants also disappear due to the same reason as for the vanishing of the integration constant c for a_1 . The coefficients of the operator $\tilde{Q}_x \tilde{Q}^{-1}$ can be computed in a similar way.

Definition. The *extended Toda hierarchy* consists of the evolutionary PDEs that are represented in the following Lax pair formalism:

$$\epsilon \frac{\partial L}{\partial t^{\beta,q}} = [A_{\beta,q}, L] := A_{\beta,q} L - L A_{\beta,q}, \quad \beta = 1, 2; \quad q \geq 0. \quad (2.24)$$

Here the operators $A_{\beta,q}$ are defined by

$$A_{1,q} = \frac{2}{q!} [L^q (\log L - c_q)]_+, \quad A_{2,q} = \frac{1}{(q+1)!} [L^{q+1}]_+, \quad (2.25)$$

and for any operator $B = \sum B_k \Lambda^k$, the operator B_+ is given by $\sum_{k \geq 0} B_k \Lambda^k$. Here the constants c_q are defined as follows

$$c_0 = 0, \quad c_q = 1 + \frac{1}{2} + \cdots + \frac{1}{q}. \quad (2.26)$$

The flows $\frac{\partial}{\partial t^{2,p}}$, $p \geq 0$, form the original Toda hierarchy (2.8). We will see in the next section that the flows $\frac{\partial}{\partial t^{1,p}}$, $p \geq 0$, coincide with those defined in [16], [33] by using a bihamiltonian recursion relation. In the literature the explicit Lax pair formalism for these flows exists only for their dispersionless limit [12], [14], [13]. To have a more concrete feeling of the form of these flows, let us write down the first three of them. By the definition (2.25), we have

$$A_{1,0} = (P\epsilon \partial_x P^{-1} - Q\epsilon \partial_x Q^{-1})_+ = \epsilon Q_x Q^{-1} = \epsilon \partial_x - \epsilon Q \partial_x Q^{-1}.$$

Since $[Q\epsilon \partial_x Q^{-1}, L] = 0$, we obtain

$$\frac{\partial w^\alpha}{\partial t^{1,0}} = w_x^\alpha, \quad \alpha = 1, 2. \quad (2.27)$$

So this first flow is just the translation along the spatial variable x . The second flow is the interpolated Toda lattice equation (2.12).

Introduce the following two operators that act on the space of smooth functions of x :

$$\begin{aligned} \mathcal{B}_+ f(x) &:= (\Lambda - 1)^{-1} \epsilon \partial_x f(x) = \sum_{k \geq 0} \frac{B_k}{k!} (\epsilon \partial_x)^k f(x), \\ \mathcal{B}_- f(x) &:= (1 - \Lambda^{-1})^{-1} \epsilon \partial_x f(x) = \sum_{k \geq 0} \frac{B_k}{k!} (-\epsilon \partial_x)^k f(x). \end{aligned} \tag{2.28}$$

Here B_k are the Bernoulli numbers. The following operator

$$A_{1,1} = (\Lambda + v)(\epsilon \partial_x - 1) + \mathcal{B}_+ v(x + \epsilon) + e^u [\epsilon \partial_x + 1 - \mathcal{B}_- u(x - \epsilon)] \Lambda^{-1} \tag{2.29}$$

(it differs from the one given by (2.25) by the operator $-L(1+Q\epsilon\partial_x Q^{-1})$ commuting with L) gives the Lax representation for the $t^{1,1}$ -flow

$$\begin{aligned} \frac{\partial v}{\partial t^{1,1}} &= vv_x + \frac{1}{\epsilon} [e^{u(x+\epsilon)} (\mathcal{B}_- u(x + \epsilon) - 2) - e^{u(x)} (\mathcal{B}_- u(x - \epsilon) - 2)], \\ \frac{\partial u}{\partial t^{1,1}} &= \frac{1}{\epsilon} [v(x)(\mathcal{B}_- u(x) - 2) - v(x - \epsilon)(\mathcal{B}_- u(x - \epsilon) - 2) \\ &\quad + \mathcal{B}_+ v(x + \epsilon) - \mathcal{B}_+ v(x - \epsilon)]. \end{aligned} \tag{2.30}$$

We finish this section with the following simple statement.

Theorem 2.2. *The components of the vector fields of the extended Toda hierarchy are homogeneous elements of the graded ring \mathcal{R} , of the degree*

$$\text{deg } \frac{\partial w^\alpha}{\partial t^{\beta,q}} = q + \mu_\beta - \mu_\alpha, \quad \alpha, \beta = 1, 2; \quad q \geq 0. \tag{2.31}$$

Here $\mu_1 = -\frac{1}{2}$, $\mu_2 = \frac{1}{2}$.

We leave the proof as an exercise for the reader.

3. BIHAMILTONIAN STRUCTURE OF THE EXTENDED TODA HIERARCHY

The existence of a bihamiltonian structure for the original Toda lattice hierarchy is well known (see for example [9], [23]). In this section we are to adopt it to the extended Toda hierarchy (2.24). The bihamiltonian structure for the original Toda hierarchy is given by the following two compatible Poisson brackets

$$\begin{aligned} \{v(x), v(y)\}_1 &= \{u(x), u(y)\}_1 = 0, \\ \{v(x), u(y)\}_1 &= \frac{1}{\epsilon} [e^{\epsilon \partial_x} - 1] \delta(x - y), \end{aligned} \tag{3.1}$$

$$\begin{aligned} \{v(x), v(y)\}_2 &= \frac{1}{\epsilon} [e^{\epsilon \partial_x} e^{u(x)} - e^{u(x)} e^{-\epsilon \partial_x}] \delta(x - y), \\ \{v(x), u(y)\}_2 &= \frac{1}{\epsilon} v(x) [e^{\epsilon \partial_x} - 1] \delta(x - y), \\ \{u(x), u(y)\}_2 &= \frac{1}{\epsilon} [e^{\epsilon \partial_x} - e^{-\epsilon \partial_x}] \delta(x - y). \end{aligned} \tag{3.2}$$

In particular, for a local Hamiltonian

$$H = \int h(w; w_x, w_{xx}, \dots; \epsilon) dx, \quad h(w; w_x, w_{xx}, \dots; \epsilon) \in \mathcal{R},$$

the Hamiltonian system with respect to the first Poisson bracket reads

$$\begin{aligned} u_t &= \{u(x), H\}_1 = \frac{1}{\epsilon} [1 - e^{-\epsilon \partial_x}] \frac{\delta H}{\delta v(x)}, \\ v_t &= \{v(x), H\}_1 = \frac{1}{\epsilon} [e^{\epsilon \partial_x} - 1] \frac{\delta H}{\delta u(x)}. \end{aligned} \tag{3.3}$$

The same Hamiltonian will generate a different PDE when the second Poisson bracket is used:

$$\begin{aligned} u_s &= \{u(x), H\}_2 = \frac{1}{\epsilon} [\Lambda - \Lambda^{-1}] \frac{\delta H}{\delta u(x)} + \frac{1}{\epsilon} [1 - \Lambda^{-1}] \frac{\delta H}{\delta v(x)}, \\ v_s &= \{v(x), H\}_2 = \frac{1}{\epsilon} v(x) [\Lambda - 1] \frac{\delta H}{\delta u(x)} + \frac{1}{\epsilon} [\Lambda e^{u(x)} - e^{u(x)} \Lambda^{-1}] \frac{\delta H}{\delta v(x)}. \end{aligned} \tag{3.4}$$

Here s is the new time variable.

We have the following main theorem of this Section:

Theorem 3.1. *The flows of the extended Toda hierarchy (2.24) are Hamiltonian systems of the form*

$$\frac{\partial w^\alpha}{\partial t^{\beta,q}} = \{w^\alpha(x), H_{\beta,q}\}_1, \quad \alpha, \beta = 1, 2; \quad q \geq 0. \tag{3.5}$$

They satisfy the following bihamiltonian recursion relation

$$\{w^\alpha(x), H_{\beta,q-1}\}_2 = \left(q + \mu_\beta + \frac{1}{2}\right) \{w^\alpha(x), H_{\beta,q}\}_1 + R_\beta^\gamma \{w^\alpha(x), H_{\gamma,q-1}\}_1. \tag{3.6}$$

Here the Hamiltonians have the form

$$H_{\beta,q} = \int h_{\beta,q}(w; w_x, \dots; \epsilon) dx, \quad \beta = 1, 2; \quad q \geq -1 \tag{3.7}$$

with the Hamiltonian densities $h_{\beta,q} = h_{\beta,q}(w; w_x, \dots; \epsilon) \in \mathcal{R}$ given by

$$h_{1,q} = \frac{2}{(q+1)!} \operatorname{res}[L^{q+1}(\log L - c_{q+1})], \quad h_{2,q} = \frac{1}{(q+2)!} \operatorname{res} L^{q+2}, \tag{3.8}$$

and

$$R_\beta^\gamma = 2\delta_2^\gamma \delta_{\beta,1}. \tag{3.9}$$

Proof. We first prove that the flows $\frac{\partial}{\partial t^{\beta,q}}$ have the Hamiltonian form (3.5). For a 1-form $\sum f_{\alpha,m}(u, u_x, \dots) du^{\alpha,m}$ on the jet space, we say that it is equivalent to zero if it is the x -derivative of another 1-form. We denote this equivalence relation by \sim . Here $w^{\alpha,m} = \frac{\partial w^\alpha(x)}{\partial x^m}$. Alternatively,

$$w^{1,m} = v^{(m)}, \quad w^{2,m} = u^{(m)}.$$

For example, we have

$$e^u dv_x + e^u u_x dv = \partial_x(e^u dv) \sim 0.$$

Under this notation, we can easily verify that

$$dh_{2,q} = \sum \frac{\partial h_{2,q}}{\partial u^{\alpha,m}} du^{\alpha,m} = \frac{1}{(q+2)!} d \operatorname{res} L^{q+2} \sim \frac{1}{(q+1)!} \operatorname{res} L^{q+1} dL \tag{3.10}$$

where $dL = dv(x) + e^{u(x)} du(x) \Lambda^{-1}$. Expand the operators $A_{\beta,q}$ that are defined in (2.25) into the form

$$A_{1,q} = \sum_{k \geq 0} a_{1,q;k} \Lambda^k, \quad A_{2,q} = \sum_{k \geq 0} a_{2,q;k} \Lambda^k, \tag{3.11}$$

then by using the definition of the Hamiltonians $H_{2,q}$ and the equivalence relation (3.10) we deduce the validity of the following identities:

$$\frac{\delta H_{2,q}}{\delta v} = a_{2,q;0}(x), \quad \frac{\delta H_{2,q}}{\delta u} = a_{2,q;1}(x - \epsilon) e^{u(x)}. \tag{3.12}$$

So from the definition of the first Poisson bracket we have

$$\begin{aligned} \{v(x), H_{2,q}\}_1 &= \frac{1}{\epsilon} (a_{2,q;1}(x) e^{u(x+\epsilon)} - a_{2,q;1}(x - \epsilon) e^{u(x)}), \\ \{u(x), H_{2,q}\}_1 &= \frac{1}{\epsilon} (a_{2,q;0}(x) - a_{2,q;0}(x - \epsilon)) \end{aligned} \tag{3.13}$$

which yields the Hamiltonian form (3.5) of the flows $\frac{\partial}{\partial t^{2,q}}$.

To prove that the flows $\frac{\partial}{\partial t^{1,q}}$ are also Hamiltonian systems with respect to the first Poisson bracket, we need first to show the validity of the following equivalence relation:

$$\text{res}(L^q d \log L) \sim \text{res}(L^{q-1} dL). \tag{3.14}$$

Indeed, from the commutativity of the operators L and $P\epsilon\partial_x P^{-1}$ we obtain

$$\begin{aligned} &d \text{res}[L^q e^{P\epsilon\partial_x P^{-1}}] \\ &\sim q \text{res}[L^{q-1} e^{P\epsilon\partial_x P^{-1}} dL] + \text{res}\left[L^q \sum_{k \geq 1} \frac{1}{(k-1)!} (P\epsilon\partial_x P^{-1})^{k-1} d(P\epsilon\partial_x P^{-1})\right] \\ &= q \text{res} L^q dL + \text{res}[L^q e^{P\epsilon\partial_x P^{-1}} d(P\epsilon\partial_x P^{-1})]. \end{aligned}$$

So from the obvious relations

$$L^q e^{P\epsilon\partial_x P^{-1}} = L^{q+1}, \quad d \text{res} L^{q+1} \sim (q+1) \text{res} L^q dL$$

we arrive at

$$\text{res}[L^q d(P\epsilon\partial_x P^{-1})] \sim \text{res} L^{q-1} dL. \tag{3.15}$$

In a similar way we obtain the following equivalence relation

$$\text{res}[L^q d(Q\epsilon\partial_x Q^{-1})] \sim -\text{res} L^{q-1} dL. \tag{3.16}$$

The equivalence relation (3.14) now readily follows from the above two equations. By using (3.14) we obtain

$$\begin{aligned} dh_{1,q} &= \frac{2}{(q+1)!} d \text{res}[L^{q+1}(\log L - c_{q+1})] \\ &\sim \frac{2}{q!} \text{res}[L^q(\log L - c_{q+1}) dL] + \frac{2}{(q+1)!} \text{res}[L^q dL] \\ &= \frac{2}{q!} \text{res}[L^q(\log L - c_q) dL]. \end{aligned} \tag{3.17}$$

It yields the following identities

$$\frac{\delta H_{1,q}}{\delta v} = a_{1,q;0}(x), \quad \frac{\delta H_{1,q}}{\delta u} = a_{1,q;1}(x - \epsilon)e^{u(x)}. \tag{3.18}$$

Here $a_{\alpha,p;k}$ are defined in (3.11). From the above identities we see that the flows $\frac{\partial}{\partial t^{1,q}}$ that is defined by (2.24) are Hamiltonian systems of the form (3.5).

We now proceed to proving the bihamiltonian recursion relation (3.6). In the case of $\alpha = 1, \beta = 2$, we can rewrite (3.6) by using the identities (3.12) into the form

$$\begin{aligned} & [\Lambda e^{u(x)} - e^{u(x)}\Lambda^{-1}]a_{2,q-1;0}(x) + v(x)[\Lambda - 1]a_{2,q-1;1}(x - \epsilon)e^{u(x)} \\ & = (q + 1)[a_{2,q;1}(x)e^{u(x+\epsilon)} - a_{2,q;1}(x - \epsilon)e^{u(x)}]. \end{aligned} \tag{3.19}$$

On the other hand, from the first and the second equality of the relation

$$(q + 1) \frac{1}{(q + 1)!} L^{q+1} = L \frac{1}{q!} L^q = \frac{1}{q!} L^q L \tag{3.20}$$

we obtain respectively the following identities

$$\begin{aligned} (q + 1)a_{2,q;1}(x) & = a_{2,q-1;0}(x + \epsilon) + v(x)a_{2,q-1;1}(x) + e^{u(x)}a_{2,q-1;2}(x - \epsilon), \\ (q + 1)a_{2,q;1}(x) & = a_{2,q-1;0}(x) + v(x + \epsilon)a_{2,q-1;1}(x) + e^{u(x+2\epsilon)}a_{2,q-1;2}(x). \end{aligned}$$

The recursion relation (3.19) can be easily verified by substituting the above two expressions of $a_{2,q;1}(x)$ into its right hand side. In the case of $\alpha = 2, \beta = 2$, the recursion relation (3.6) can be also verified by using the identities in (3.20). Finally, for the case of $\beta = 1$ the recursion relation (3.6) follows from the following trivial identities

$$\begin{aligned} q \frac{2}{q!} L^q (\log L - c_q) & = L \frac{2}{(q - 1)!} L^{q-1} (\log L - c_{q-1}) - 2 \frac{1}{q!} L^q \\ & = \frac{2}{(q - 1)!} L^{q-1} (\log L - c_{q-1})L - 2 \frac{1}{q!} L^q. \end{aligned}$$

The theorem is proved. □

In [16], [33] an extended Toda hierarchy was defined by using the bihamiltonian recursion relation (3.6), and the Hamiltonians are defined implicitly from this recursion relation. The above theorem shows that this extended Toda hierarchy coincides with the one that is defined by (2.24), it also gives an explicit expression of the densities of the Hamiltonians of the hierarchy. We list here the first few of them

$$\begin{aligned} h_{1,-1} & = \mathcal{B}_- u(x), \quad h_{2,-1} = v(x), \\ h_{1,0} & = \mathcal{B}_+(v(x) + v(x + \epsilon)) - 2v(x) + v(x) \mathcal{B}_- u(x), \\ h_{2,0} & = v(x)^2 + e^{u(x)} + e^{u(x+\epsilon)}, \end{aligned} \tag{3.21}$$

where the operators \mathcal{B}_\pm are defined in (2.28). We will see below that these densities of the Hamiltonians possess an important symmetry property which will be used to define the tau functions for solutions of the extended Toda hierarchy.

4. TAU FUNCTIONS FOR THE EXTENDED TODA HIERARCHY

We now proceed to define the tau functions for solutions of the extended Toda hierarchy. Denote by $\tilde{\mathcal{R}}$ the subset of homogeneous elements of the ring \mathcal{R} , i.e., elements of the form

$$f = \sum_{k \geq 0} f_k(w, w_x, \dots) \epsilon^k, \quad \sum_{i=1}^2 \sum_{m \geq 1} m w^{i,m} \frac{\partial f_k}{\partial w^{i,m}} = k f_k,$$

where f_k are homogeneous polynomials of $e^{\pm u}$, $v^{(m)}$, $u^{(m)}$ for $m \geq 0$ with $\deg f_k = \deg f - k$. From the definition of the extended Toda hierarchy (2.24) and the densities of the hamiltonian (3.8) we know that $\epsilon \frac{\partial w^\alpha}{\partial t^{\beta,q}}$, $h_{\beta,q} \in \tilde{\mathcal{R}}$. The degrees of the flows are given in (2.31) and the degrees of $h_{\beta,q}$ are given by

$$\deg h_{\beta,q} = q + \frac{3}{2} + \mu_\beta.$$

Lemma 4.1. *The following formulae hold true:*

$$\frac{\partial \log L}{\partial t^{\beta,q}} = [A_{\beta,q}, \log L], \quad \beta = 1, 2; \quad q \geq 0. \tag{4.1}$$

Proof. From (2.18) we have

$$\left[\frac{\partial(P\epsilon\partial_x P^{-1})}{\partial t^{\beta,q}}, L^m \right] + [P\epsilon\partial_x P^{-1}, [A_{\beta,q}, L^m]] = 0.$$

The Jacobi identity and the commutativity between the operators L and $P\epsilon\partial_x P^{-1}$ then imply the following identity

$$\left[\frac{\partial(P\epsilon\partial_x P^{-1})}{\partial t^{\beta,q}} - [A_{\beta,q}, P\epsilon\partial_x P^{-1}], L^m \right] = 0.$$

Since the operator $\frac{\partial(P\epsilon\partial_x P^{-1})}{\partial t^{\beta,q}} - [A_{\beta,q}, P\epsilon\partial_x P^{-1}]$ has the form $\sum_{k \geq 1} f_k \Lambda^{-k}$ with coefficients f_k being elements of $\tilde{\mathcal{R}}$, we obtain from the last equality the formula

$$\frac{\partial(P\epsilon\partial_x P^{-1})}{\partial t^{\beta,q}} - [A_{\beta,q}, P\epsilon\partial_x P^{-1}] = 0.$$

In a similar way we can also get the formula

$$\frac{\partial(Q\epsilon\partial_x Q^{-1})}{\partial t^{\beta,q}} - [A_{\beta,q}, Q\epsilon\partial_x Q^{-1}] = 0.$$

So the lemma follows from the definition of $\log L$ and from the last two identities. □

We introduce now the functions $\Omega_{\alpha,p;\beta,q}$ by the formula

$$(\Lambda - 1)\Omega_{\alpha,p;\beta,q} := \epsilon \frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} = \begin{cases} \frac{2}{p!} \operatorname{res}[A_{\beta,q}, L^p(\log L - c_p)], & \alpha = 1; \\ \frac{1}{(p+1)!} \operatorname{res}[A_{\beta,q}, L^{p+1}], & \alpha = 2, \end{cases} \tag{4.2}$$

and by the homogeneity condition

$$\Omega_{\alpha,p;\beta,q} \in \tilde{\mathcal{R}}, \quad \deg \Omega_{\alpha,p;\beta,q} = p + q + 1 + \mu_\alpha + \mu_\beta, \quad \alpha, \beta = 1, 2; \quad p, q \geq 0. \tag{4.3}$$

Note that in the above definition the second equality of (4.2) follows from the definition (2.24), (3.8) and from the above lemma. The right hand side is a total

x -derivative of a homogeneous element in \mathcal{R} . Therefore $\Omega_{\alpha,p;\beta,q} \in \tilde{\mathcal{R}}$ and the conditions (4.2) and (4.3) specify $\Omega_{\alpha,p;\beta,q}$ uniquely. The only exception is $\Omega_{1,0;1,0}$ that should be a homogeneous element of the degree 0. This is set to be

$$\Omega_{1,0;1,0} = \mathcal{B}_- \mathcal{B}_+ u.$$

The following theorem shows that $\Omega_{\alpha,p;\beta,q}$ is symmetric with respect to the pair of its indices (α, p) and (β, q) :

Theorem 4.2. *The extended Toda hierarchy has the following tau-symmetry property:*

$$\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} = \frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}}, \quad \alpha, \beta = 1, 2; p, q \geq 0. \tag{4.4}$$

Proof. Let us prove the theorem for the case when $\alpha = 1, \beta = 2$, other cases are proved in a similar way. From the second identity of (4.2) we obtain

$$\begin{aligned} \frac{\partial h_{1,p-1}}{\partial t^{2,q}} &= \frac{2}{p!(q+1)!} \operatorname{res}[(L^{q+1})_+, L^p(\log L - c_p)] \\ &= \frac{2}{p!(q+1)!} \operatorname{res}[-(L^{q+1})_-, L^p(\log L - c_p)] \\ &= \frac{2}{p!(q+1)!} \operatorname{res}[(L^p(\log L - c_p))_+, (L^{q+1})_-] \\ &= \frac{2}{p!(q+1)!} \operatorname{res}[(L^p(\log L - c_p))_+, L^{q+1}] = \frac{\partial h_{2,q-1}}{\partial t^{1,p}}. \end{aligned} \tag{4.5}$$

Theorem is proved. □

From the above theorem and the definition (4.2) it follows that $\frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial t^{\sigma,k}}$ is symmetric with respect to the three pairs of indices $(\alpha, p), (\beta, q), (\sigma, k)$. This property justifies the following definition of tau function for the extended Toda hierarchy:

Definition. For any solution of the extended Toda hierarchy there exists a function τ of the spatial and time variables $x, t^{\alpha,p}, \alpha = 1, 2, p \geq 0$, and of ϵ such that

$$\Omega_{\alpha,p;\beta,q} = \epsilon^2 \frac{\partial^2 \log \tau}{\partial t^{\alpha,p} \partial t^{\beta,q}} \tag{4.6}$$

hold true for any $\alpha, \beta = 1, 2, p, q \geq 0$.

Recall that the solutions considered in this paper are assumed to be formal power series in ϵ .

Since the first flow $\frac{\partial}{\partial t^{1,0}}$ of the extended Toda hierarchy coincides with the translation along the spatial variable x , we can modify the above definition of the tau function by requiring that

$$\frac{\partial \log \tau}{\partial t^{1,0}} = \frac{\partial \log \tau}{\partial x}. \tag{4.7}$$

Corollary 4.3. *The densities of the Hamiltonians of the extended Toda hierarchy have the following expressions in terms of the tau function:*

$$h_{\alpha,p} = \epsilon(\Lambda - 1) \frac{\partial \log \tau}{\partial t^{\alpha,p+1}}, \quad \alpha = 1, 2; p \geq -1. \tag{4.8}$$

Proof. From the definition of $\Omega_{\alpha,p;1,0}$ we get

$$h_{\alpha,p-1} = \sum_{k \geq 1} \frac{\epsilon^{k-1}}{k!} \partial_x^{k-1} \Omega_{\alpha,p;1,0} = \sum_{k \geq 1} \frac{\epsilon^{k+1}}{k!} \partial_x^{k-1} \frac{\partial^2 \log \tau}{\partial t^{\alpha,p} \partial t^{1,0}} = \epsilon(\Lambda - 1) \frac{\partial \log \tau}{\partial t^{\alpha,p}}.$$

Here we used (4.7). The corollary is proved. □

Our notion of tau function for the extended Toda hierarchy follows that of Date, Jimbo, Kashiwara and Miwa designed for the KP hierarchy [3]. Note that the above corollary implies in particular the following relations of the dependent variables v, u of the extended Toda hierarchy with the tau function:

$$v = \epsilon \frac{\partial}{\partial t^{2,0}} \log \frac{\tau(x + \epsilon)}{\tau(x)}, \quad u = \log \frac{\tau(x + \epsilon)\tau(x - \epsilon)}{\tau^2(x)}. \tag{4.9}$$

In this formula we omit the dependence of the tau function on all the times $t^{\alpha,p}$ but the very first one $t^{1,0} = x$.

Remark 1. The above formulae mean that, the dependent variables u, v of the extended Toda hierarchy are not normal coordinates in the sense of [7]. Because of this the relationships between the tau-function and the Hamiltonian densities in the present paper look more complicated than in the general setting of [7].

If we return back to the variable q_n of the original Toda lattice equation (1.1), then from the above relation we have

$$q_n = \log \frac{\tau(n)}{\tau(n-1)}. \tag{4.10}$$

So the tau function for the extended Toda hierarchy also agrees with the function that was introduced by Hirota and Satsuma [19] to convert the Toda lattice equation into a bilinear form. We will discuss the bilinear formulation of the extended Toda hierarchy in a separate publication.

Remark 2. The dispersionless limit $\epsilon \rightarrow 0$ of the bihamiltonian structure (3.1), (3.2) coincides with the canonical Poisson pencil on the loop space $\mathcal{L}(M)$ of the Frobenius manifold $M = M_{\tilde{W}^{(1)}(A_1)}$ constructed in [10] on the orbit space of the extended affine Weyl group $\tilde{W}^{(1)}(A_1)$. The Frobenius manifolds $M_{\tilde{W}^{(k)}(A_{k+m-1})}$ on the orbit spaces of more general extended affine Weyl groups $\tilde{W}^{(k)}(A_{k+m-1})$ of the A -series are obtained by the dispersionless limits of extended Toda-like systems associated with the difference Lax operators of the form

$$L = \Lambda^k + a_1(x)\Lambda^{k-1} + \dots + a_{k+m}(x)\Lambda^{-m}, \quad a_{k+m}(x) \neq 0.$$

This extended hierarchy coincides with the one associated with the Frobenius manifold $M_{\tilde{W}^{(k)}(A_{k+m-1})}$ according to the general scheme of [7]. We will give details in a separate publication. Recall that, in [10] there were also constructed Frobenius manifolds on the orbit spaces of extended affine Weyl groups associated with the Dynkin diagrams of the $B C D E F G$ series. At the moment we do not know how to construct Lax representation of the integrable hierarchies associated, according to the results of [7], with these Frobenius manifolds. We plan to study this problem in subsequent publications.

5. AN ALTERNATIVE REPRESENTATION OF THE EXTENDED TODA HIERARCHY — THE EXTENDED NLS HIERARCHY

In this section we present an alternative representation of the extended Toda hierarchy that is defined in the Section 2. We are to choose $t^{2,0}$ as spatial variable and write down the evolutionary PDEs that are satisfied by the functions u, v under this new spatial variable. Let us redenote the time variables as follows:

$$T^{1,p} = t^{2,p}, \quad T^{2,p} = t^{1,p}, \quad p \geq 0, \tag{5.1}$$

and specify $X = T^{1,0}$ as the spatial variable. For the convenience of presentation, we use the following quantities as the dependent variables:

$$\tilde{w}^1(X, T) \equiv \varphi = v(x - \epsilon), \quad \tilde{w}^2(X, T) \equiv \rho = e^{u(x)}. \tag{5.2}$$

In terms of the tau function of the extended Toda hierarchy, these new dependent variables have the expression

$$\varphi = \epsilon(1 - \Lambda^{-1}) \frac{\partial \log \tau}{\partial T^{1,0}}, \quad \rho = \exp[(1 - \Lambda^{-1})(\Lambda - 1) \log \tau]. \tag{5.3}$$

Let us first proceed to writing down the Lax pair formalism for the hierarchy that is satisfied by $\varphi(X, T), \rho(X, T)$. Note that the extended Toda hierarchy is the compatibility condition of the following linear systems

$$L\psi = \lambda\psi, \tag{5.4}$$

$$\frac{\partial \psi}{\partial t^{\alpha,p}} = \epsilon^{-1} A_{\alpha,p} \psi, \tag{5.5}$$

where $A_{\alpha,p}$ are defined in (2.25) and λ is the spectral parameter. By using the equation $\epsilon \partial_{t^{2,0}} \psi = (\Lambda + v)\psi$, we can rewrite the linear system (5.4) in the form

$$\mathcal{L} \psi = \lambda \psi \tag{5.6}$$

with the operator \mathcal{L} defined by

$$\mathcal{L} = \epsilon \partial_X + \rho(\epsilon \partial_X - \varphi)^{-1}. \tag{5.7}$$

Here the pseudo-differential operator $(\epsilon \partial_X - \varphi)^{-1}$ has the expansion

$$(\epsilon \partial_X - \varphi)^{-1} = \sum_{k \geq 1} a_k (\epsilon \partial_X)^{-k} \tag{5.8}$$

and the coefficients a_k are uniquely defined by the relation

$$(\epsilon \partial_X - \varphi) \left(\sum_{k \geq 1} a_k (\epsilon \partial_X)^{-k} \right) = 1. \tag{5.9}$$

For example, we have

$$a_1 = 1, \quad a_2 = \varphi, \quad a_3 = -\epsilon \varphi_X + \varphi^2.$$

We can also reexpress the operators $A_{\alpha,p}$ as differential operators in $\epsilon \partial_X$. This can be easily done by the substitution

$$\Lambda^k \mapsto (\epsilon \partial_X - \varphi(x + k\epsilon)) \dots (\epsilon \partial_X - \varphi(x + \epsilon)). \tag{5.10}$$

So the linear systems in (5.5) can be expressed in the form

$$\frac{\partial \psi}{\partial T^{\alpha,p}} = \epsilon^{-1} \mathcal{A}_{\alpha,p} \psi \tag{5.11}$$

with

$$\mathcal{A}_{1,p} = \frac{1}{(p+1)!} (\mathcal{L}^{p+1})_+, \quad \mathcal{A}_{2,p} = \frac{2}{p!} (\mathcal{L}^p (\log \mathcal{L} - c_p))_+, \tag{5.12}$$

the subscript + here means to take the differential part of a pseudo-differential operator. The pseudo-differential operator $\log \mathcal{L}$ is obtained from $\log L$ by the substitution of (5.10) and

$$\Lambda^{-k} \mapsto (\epsilon \partial_X - \varphi(x - (k-1)\epsilon))^{-1} \dots (\epsilon \partial_X - \varphi(x))^{-1}. \tag{5.13}$$

The coefficients of $\log \mathcal{L}$ can be expressed in terms of the new dependent variables φ, ρ and their X -derivatives. This can be achieved by using the system of equations (2.12) to express $\frac{\partial^m \tilde{w}^\alpha}{\partial x^m}$, $m \geq 1$, in terms of φ, ρ and their X -derivatives. For example, we have

$$\frac{\partial \varphi}{\partial x} = \partial_X \left[\log \rho + \frac{\epsilon^2}{12\rho^3} (\rho \rho_{XX} - \rho_X^2 - \rho \varphi_X^2) + \mathcal{O}(\epsilon^4) \right], \tag{5.14}$$

$$\frac{\partial \rho}{\partial x} = \partial_X \left[\varphi - \frac{\epsilon^2}{6\rho^2} (\rho \varphi_{XX} - \varphi_X \rho_X) + \mathcal{O}(\epsilon^4) \right]. \tag{5.15}$$

Now the compatibility condition of the linear systems (5.6), (5.11) takes the form

$$\epsilon \frac{\partial \mathcal{L}}{\partial T^{\alpha,p}} = [\mathcal{A}_{\alpha,p}, \mathcal{L}], \quad \alpha = 1, 2; p \geq 0. \tag{5.16}$$

The $T^{1,0}$ -flow coincides with the shift along X , and the $T^{2,0}$ -flow is given by (5.14) and (5.15). The $T^{1,1}$ -flow has the form

$$\begin{aligned} \frac{\partial \varphi}{\partial T^{1,1}} &= \partial_X (-\epsilon \varphi_X + \varphi^2 + 2\rho), \\ \frac{\partial \rho}{\partial T^{1,1}} &= \partial_X (\epsilon \rho_X + 2\varphi \rho). \end{aligned} \tag{5.17}$$

This integrable system appears in the study of nonlinear water waves in [2], [20]. In terms of the new variables

$$q = e^{\epsilon^{-1} \partial_X^{-1} v} = \rho e^{\epsilon^{-1} \partial_X^{-1} \varphi}, \quad r = e^u e^{-\epsilon^{-1} \partial_X^{-1} v} = e^{-\epsilon^{-1} \partial_X^{-1} \varphi}, \tag{5.18}$$

or, equivalently,

$$\rho = q r, \quad \varphi = -\epsilon \frac{r_X}{r},$$

the above system takes the form

$$\frac{\partial q}{\partial T^{1,1}} = \epsilon q_{XX} + 2\epsilon^{-1} q^2 r, \quad \frac{\partial r}{\partial T^{1,1}} = -\epsilon r_{XX} - 2\epsilon^{-1} q r^2. \tag{5.19}$$

These functions have the the following simple expressions in terms of the tau function of the extended Toda hierarchy:

$$q = \frac{\tau(x + \epsilon)}{\tau(x)}, \quad r = \frac{\tau(x - \epsilon)}{\tau(x)}. \tag{5.20}$$

Under the constraints $\epsilon = i$, $r = \pm q^*$ the system (5.19) is reduced to the well known nonlinear Schrödinger equation (NLS) [32]. Due to this fact, we will call the hierarchy (5.16) the *extended NLS hierarchy*.

The extended NLS hierarchy also possesses a bihamiltonian structure. The related compatible Poisson brackets are given by

$$\begin{aligned} \{\varphi(X), \varphi(Y)\}_1 &= \{\rho(X), \rho(Y)\}_1 = 0, \\ \{\varphi(X), \rho(Y)\}_1 &= \delta'(X - Y); \end{aligned} \tag{5.21}$$

$$\begin{aligned} \{\varphi(X), \varphi(Y)\}_2 &= 2\delta'(X - Y), \\ \{\varphi(X), \rho(Y)\}_2 &= \varphi(X)\delta'(X - Y) + \varphi_X\delta(X - Y) - \epsilon\delta''(X - Y), \\ \{\rho(X), \rho(Y)\}_2 &= [\rho(X)\partial_X + \partial_X\rho(X)]\delta(X - Y). \end{aligned} \tag{5.22}$$

This Poisson pencil was given in [1] for the bihamiltonian structure of the system (5.17). It is easy to verify that the extended NLS hierarchy hierarchy (5.16) has the Hamiltonian form

$$\frac{\partial \tilde{w}^\alpha}{\partial T^{\beta,q}} = \{\tilde{w}^\alpha(X), \tilde{H}_{\beta,q}\}_1, \quad \alpha, \beta = 1, 2; \quad q \geq 0. \tag{5.23}$$

Here the Hamiltonians $\tilde{H}_{\beta,q} = \int \tilde{h}_{\beta,q} dX$ are defined by

$$\tilde{h}_{1,q} = \frac{1}{(q+2)!} \operatorname{res} \mathcal{L}^{q+2}, \quad \tilde{h}_{2,q} = \frac{2}{(q+1)!} \operatorname{res}[\mathcal{L}^{q+1}(\log \mathcal{L} - c_{q+1})] \tag{5.24}$$

and the residue of a pseudo-differential operator equals the coefficient of ∂_X^{-1} . The hierarchy satisfies the following bihamiltonian recursion relation:

$$\begin{aligned} \{\tilde{w}^\alpha(X), \tilde{H}_{\beta,q-1}\}_2 &= \left(q + \frac{1}{2} + \tilde{\mu}_\beta\right) \{\tilde{w}^\alpha(X), \tilde{H}_{\beta,q}\}_1 + \tilde{R}_\beta^\gamma \{\tilde{w}^\alpha(X), \tilde{H}_{\gamma,q-1}\}_1, \\ &\alpha, \beta = 1, 2; \quad q \geq 0. \end{aligned} \tag{5.25}$$

Here $\tilde{\mu}_1 = -\tilde{\mu}_2 = \frac{1}{2}$, $\tilde{R}_\beta^\alpha = 2\delta_1^\alpha \delta_{\beta 2}$.

Remark. In the “dispersionless” limit $\epsilon \rightarrow 0$ the substitution (5.2) becomes

$$\varphi = v, \quad \rho = e^u. \tag{5.26}$$

This coincides with the Legendre type transformation S_2 of [6] (see Appendix B) transforming the Frobenius manifold associated with Toda lattice with the potential

$$F_{\text{Toda}} = \frac{1}{2}v^2u + e^u$$

to the Frobenius manifold associated with NLS with the potential

$$F_{\text{NLS}} = \frac{1}{2}\varphi^2\rho + \frac{1}{2}\rho^2 \left[\log \rho - \frac{3}{2} \right]$$

(see Example B.1 in [6]). It looks plausible that the trick similar to the above one will work also for an arbitrary semisimple Frobenius manifold in order to lift the Legendre-type transforms of the Frobenius manifold to a transformation of the integrable hierarchy associated with this manifold. We will describe these transformations in a separate publication.

6. THE EXTENDED TODA HIERARCHY AND THE $\mathbb{C}P^1$ TOPOLOGICAL SIGMA MODEL

Let $\phi_1 = 1 \in H^0(\mathbb{C}P^1)$, $\phi_2 = \omega \in H^2(\mathbb{C}P^1)$ be the two primary fields for the $\mathbb{C}P^1$ topological sigma model. The 2-form ω is assumed to be normalized by the condition

$$\int_{\mathbb{C}P^1} \omega = 1.$$

The free energy of the $\mathbb{C}P^1$ topological sigma-model is a function of infinite number of *coupling parameters*

$$\mathbf{t} = (t^{1,0}, t^{2,0}, t^{1,1}, t^{2,1}, \dots)$$

and of ϵ defined by the following genus expansion form:

$$\mathcal{F}(\mathbf{t}; \epsilon) = \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g(\mathbf{t}). \tag{6.1}$$

The parameter ϵ is called here the string coupling constant, and the function $\mathcal{F}_g = \mathcal{F}_g(\mathbf{t})$ is called the genus g free energy which is given by

$$\mathcal{F}_g = \sum \frac{1}{m!} t^{\alpha_1, p_1} \dots t^{\alpha_m, p_m} \langle \tau_{p_1}(\phi_{\alpha_1}) \dots \tau_{p_m}(\phi_{\alpha_m}) \rangle_g, \tag{6.2}$$

where $\tau_p(\phi_\alpha)$ are the gravitational descendent of the primary fields with coupling constants $t^{\alpha, p}$, and the rational numbers $\langle \tau_{p_1}(\phi_{\alpha_1}) \dots \tau_{p_m}(\phi_{\alpha_m}) \rangle_g$ are given by the genus g Gromov–Witten invariants and their descendants of $\mathbb{C}P^1$:

$$\begin{aligned} & \langle \tau_{p_1}(\phi_{\alpha_1}) \dots \tau_{p_m}(\phi_{\alpha_m}) \rangle_g \\ &= \sum_{\beta} q^{\beta} \int_{[\bar{M}_{g,m}(\mathbb{C}P^1, \beta)]^{\text{virt}}} \text{ev}_1^* \phi_{\alpha_1} \wedge \psi_1^{p_1} \wedge \dots \wedge \text{ev}_m^* \phi_{\alpha_m} \wedge \psi_m^{p_m}. \end{aligned}$$

Here $\bar{M}_{g,m}(\mathbb{C}P^1, \beta)$ is the moduli space of stable curves of genus g with m markings of the given degree $\beta \in H_2(\mathbb{C}P^1; \mathbb{Z})$, ev_i is the evaluation map

$$\text{ev}_i: \bar{M}_{g,m}(\mathbb{C}P^1, \beta) \rightarrow \mathbb{C}P^1$$

corresponding to the i -th marking, ψ_i is the first Chern class of the tautological line bundle over the moduli space corresponding to the i -th marking. According to the divisor axiom [22] the dependence of the Gromov–Witten potential on the indeterminate q appears only through the combination $q e^{t^{2,0}}$. We will therefore omit the dependence on q in the formulae.

The conjectural relation of the $\mathbb{C}P^1$ topological sigma model with the extended Toda hierarchy can now be stated in a similar way as the Kontsevich–Witten result [31], [21], [30]² does for the relation of the 2d topological gravity with the KdV hierarchy. Namely,

Theorem 6.1 (Toda conjecture). *The functions*

$$\begin{aligned} u(x, \mathbf{t}; \epsilon) &= \mathcal{F}(t^{1,0} + x + \epsilon) - 2\mathcal{F}(t^{1,0} + x) + \mathcal{F}(t^{1,0} + x - \epsilon), \\ v(x, \mathbf{t}; \epsilon) &= \epsilon \frac{\partial}{\partial t^{2,0}} [\mathcal{F}(t^{1,0} + x + \epsilon) - \mathcal{F}(t^{1,0} + x)] \end{aligned} \tag{6.3}$$

²An alternative proof was given recently by Okounkov in [25].

satisfy the equations of the extended Toda hierarchy (2.24). In these formulae we write explicitly down only those arguments of the function \mathcal{F} that have been modified. This particular solution is uniquely specified by the string equation

$$\sum_{p \geq 1} t^{\alpha,p} \frac{\partial \mathcal{F}}{\partial t^{\alpha,p-1}} + \frac{1}{\epsilon^2} t^{1,0} t^{2,0} = \frac{\partial \mathcal{F}}{\partial t^{1,0}}. \tag{6.4}$$

The bihamiltonian description of the extended Toda hierarchy obtained in Section 3 above along with the tau-structure described in Section 4 enables one to rewrite the bihamiltonian recursion (3.6) in the form of a recursion for the correlators of the $\mathbb{C}P^1$ topological sigma-model. Namely, let us introduce, following [31], the functions $\langle\langle \tau_p(\phi_\alpha) \tau_q(\phi_\beta) \dots \rangle\rangle$ of \mathbf{t}, ϵ by

$$\langle\langle \tau_{p_1}(\phi_{\alpha_1}) \dots \tau_{p_m}(\phi_{\alpha_m}) \rangle\rangle = \epsilon^m \frac{\partial}{\partial t^{\alpha_1, p_1}} \dots \frac{\partial}{\partial t^{\alpha_m, p_m}} \mathcal{F}(\mathbf{t}; \epsilon). \tag{6.5}$$

Then the following recursion relations hold true

$$(n + 1)(\Lambda - 1) \langle\langle \tau_n(\omega) \rangle\rangle = v(\Lambda - 1) \langle\langle \tau_{n-1}(\omega) \rangle\rangle + (\Lambda + 1) \langle\langle \tau_0(\omega) \tau_{n-1}(\omega) \rangle\rangle, \tag{6.6}$$

$$n(\Lambda - 1) \langle\langle \tau_n(1) \rangle\rangle = v(\Lambda - 1) \langle\langle \tau_{n-1}(1) \rangle\rangle - 2(\Lambda - 1) \langle\langle \tau_{n-1}(\omega) \rangle\rangle + (\Lambda + 1) \langle\langle \tau_0(\omega) \tau_{n-1}(1) \rangle\rangle. \tag{6.7}$$

In these recursion relations

$$\Lambda = \exp \epsilon \frac{\partial}{\partial t^{1,0}}, \quad v = \epsilon(\Lambda - 1) \frac{\partial \mathcal{F}}{\partial t^{2,0}}.$$

We are to emphasize that, these recursion relations hold true for an arbitrary solution of extended Toda hierarchy if one defines the ‘‘correlators’’ by the equation (6.5) with the function \mathcal{F} corresponding to the logarithm of the tau function of this solution³. The needed solution is specified by (6.6), (6.7) together with the string equation. In this case the recursion relations describe the topology of the forgetting map [22]

$$\bar{M}_{g,n}(\mathbb{C}P^1) \rightarrow \bar{M}_{g,n-1}(\mathbb{C}P^1).$$

Due to the discussion of the last Section, we can equally state Theorem 6.1 as follows. The free energy (6.1) is the logarithm of a particular tau function of the extended NLS hierarchy (5.16).

The proof of Theorem 6.1 at the genus one approximation can be obtained using results of [5], [9], [12], [14], [13], [33], see also important papers [16], [27], [26]. The crucial point in proving the validity of this conjecture in full genera is the Givental’s result on the Virasoro constraints for $\mathbb{C}P^1$ [18], [17]. Probably, one can derive our Toda conjecture from the results of Okounkov and Pandharipande [27], [26] using the arguments of Getzler’s paper [16] along with the Givental’s result. From our point of view the most natural way of proving the Conjecture is that to use the properties of the Virasoro symmetries of the extended Toda hierarchy and the uniqueness of solution of the loop equation [11], [7]. The details of the proof along with the construction of the Virasoro symmetries of the extended Toda hierarchy will be published in a separate paper (see [8]).

³In the literature sometimes these recursion relations together with the Toda equations (2.12) are called Toda conjecture.

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G. C., B. D.: INTERNATIONAL SCHOOL FOR ADVANCED STUDY (SISSA), VIA BEIRUT 2–4, 34014 TRIESTE, ITALY

E-mail address: carlet@sisssa.it, gcar8721@usyd.edu.au

E-mail address: dubrovin@sisssa.it

Y. Zh.: DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING 100084, P.R.CHINA

E-mail address: youjin@mail.tsinghua.edu.cn