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EXTENDED TODA LATTICE

G. Carlet*

We introduce nonlocal flows that commute with those of the classical Toda hierarchy. We define a logarithm of the difference Lax operator and use it to obtain a Lax representation of the new flows.

Keywords: Toda lattice, logarithm of a difference operator, Lax representation, bi-Hamiltonian formalism

The Toda lattice equation is a nonlinear evolution equation introduced by Toda [2] describing an infinite system of masses on a line that interact through an exponential force. In suitable coordinates, it can be written as the system

$$\begin{aligned} \frac{\partial}{\partial t}u_n &= e^{v_{n+1}} - e^{v_n}, \\ \frac{\partial}{\partial t}v_n &= u_n - u_{n-1}, \end{aligned} \tag{1}$$

where $n \in \mathbb{Z}$. It was soon realized that this equation is completely integrable, i.e., it admits infinite conserved quantities, can be solved for rapidly decreasing boundary conditions through the inverse scattering transform [3], and admits explicit quasiperiodic solutions by algebro-geometric methods [4]. It has found important applications in many different fields, in particular, in the theory of Gromov–Witten invariants of $\mathbb{C}P^1$, where the present extension plays a particular role [5].

The Toda lattice equation can be seen as the first element of a whole hierarchy of commuting flows, the Toda lattice hierarchy. We can write all the flows using the Lax representation [3]

$$\epsilon \frac{\partial L}{\partial t_q} = \left[\frac{1}{(q+1)!}(L^{q+1})_+, L\right],$$

where L is the difference operator

$$L = \Lambda + u + e^v \Lambda^{-1}.$$

Here we use a notation (see [6]) with a continuous space variable $x = n\epsilon$, where ϵ is the lattice spacing and $t_0 = \epsilon t$, i.e., $u(x) = u_n$ and $v(x) = v_n$ for $x = \epsilon n$. We let Λ denote the shift operator $\Lambda f(x) = f(x+\epsilon)$. Given any difference operator $A = \sum_{k\in\mathbb{Z}} a_k \Lambda^k$, we let A_+ and A_- denote the respective positive and negative parts, i.e., $A_+ = \sum_{k\geq 0} a_k \Lambda^k$, $A = A_+ + A_-$. The first flow t_0 is simply given by Toda equations (1); as a further example, the second flow is given by

$$\begin{aligned} \epsilon u_{t_1}(x) &= \frac{1}{2}\big((u(x+\epsilon) + u(x))e^{v(x+\epsilon)} - (u(x-\epsilon) + u(x))e^{v(x)}\big), \\ \epsilon v_{t_1}(x) &= \frac{1}{2}\big(e^{v(x+\epsilon)} - e^{v(x-\epsilon)} + u^2(x) - u^2(x-\epsilon)\big). \end{aligned}$$

*SISSA, Via Beirut 2-4, 34014 Trieste, Italy, e-mail: carlet@sissa.it.

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This hierarchy is Hamiltonian with respect to two different Hamiltonian structures [7]. In our continuous formulation, the first is given by the Poisson bracket

$$\begin{aligned} \{u(x), v(y)\}_1 &= \frac{1}{\epsilon}(\delta(x-y+\epsilon) - \delta(x-y)), \\ \{u(x), u(y)\}_1 &= \{v(x), v(y)\}_1 = 0. \end{aligned} \tag{2}$$

The second Poisson bracket is

$$\begin{aligned} \{u(x), u(y)\}_2 &= \frac{1}{\epsilon}(e^{v(x+\epsilon)}\delta(x-y+\epsilon) - e^{v(x)}\delta(x-y-\epsilon)), \\ \{u(x), v(y)\}_2 &= \frac{1}{\epsilon}u(x)(\delta(x-y+\epsilon) - \delta(x-y)), \\ \{v(x), v(y)\}_2 &= \frac{1}{\epsilon}(\delta(x-y+\epsilon) - \delta(x-y-\epsilon)). \end{aligned} \tag{3}$$

The equations of motion can be written as

$$\frac{d}{dt_q}\cdot = \{\,\cdot\,, \bar{h}_q\}_1 = \frac{1}{q+1}\{\,\cdot\,, \bar{h}_{q-1}\}_2, \tag{4}$$

where $\bar{h}_q = \int h_q\, dx$ and the Hamiltonians are given as traces of powers of the Lax operator L,

$$h_q = \frac{1}{(q+2)!}\operatorname{Res}(L^{q+2}). \tag{5}$$

Given any difference operator $A = \sum_{k\in\mathbb{Z}} a_k\Lambda^k$, its residue is defined by $\operatorname{Res} A = a_0$.

The presence of a two Hamiltonian structures permits obtaining all the Hamiltonians through Lenard–Magri [8] recursive relation (4) starting from a Casimir of the first Poisson bracket. In our case, if we start from the Casimir $h_{-1} = u$, we obtain all the Hamiltonians defined above. Moreover, this procedure guarantees that all the resulting Hamiltonians commute among themselves.

To see why and how this hierarchy of equations could be extended, we consider its dispersionless limit that is obtained by setting $\epsilon \to 0$. It can be shown [6] that the Lax representation of the dispersionless flows is given by

$$\frac{\partial \mathcal{L}}{\partial t_q} = \left\{\frac{1}{(q+1)!}(\mathcal{L}^{q+1})_+, \mathcal{L}\right\}.$$

In this case, $\mathcal{L}$ is a function of x and of the additional variable p,

$$\mathcal{L} = p + u(x) + e^{v(x)}p^{-1},$$

and the bracket is

$$\{\mathcal{B}, \mathcal{C}\} = p\frac{\partial \mathcal{B}}{\partial p}\frac{\partial \mathcal{C}}{\partial x} - p\frac{\partial \mathcal{C}}{\partial p}\frac{\partial \mathcal{B}}{\partial x},$$

where $\mathcal{B}$ and $\mathcal{C}$ are functions of p and x and $(\mathcal{B})_+$ means that only nonnegative powers of p are considered in the power series expansion of $\mathcal{B}$.

The dispersionless Hamiltonians and Poisson brackets are simply obtained from their dispersive counterparts (5) and (2), (3) by setting $\epsilon \to 0$. In particular, the same recursive relation (4) as above holds in the dispersionless case.

In [9], it was noted that for the genus-zero approximation of the topological $\mathbb{C}P^1$ model, new flows can be added to the usual dispersionless flows given above; their Lax representation is

$$\frac{\partial \mathcal{L}}{\partial \tilde{t}_q} = \left\{ \frac{2}{q!} (\mathcal{L}^q (\log \mathcal{L} - c_q))_+, \mathcal{L} \right\}, \tag{6}$$

where $c_q = \sum_{k=1}^{q} (1/k)$, $c_0 = 0$. The logarithm of $\mathcal{L}$ must be understood as

$$\log \mathcal{L} = \frac{1}{2} v + \frac{1}{2} \log(1 + u p^{-1} + e^v p^{-2}) + \frac{1}{2} \log(1 + u e^{-v} p + e^{-v} p^2),$$

where the first logarithm in the r.h.s. is seen as an expansion in negative powers of p and the second one in positive powers of p.

These flows can be expressed in Hamiltonian form by

$$\frac{d}{d\tilde{t}_q} \cdot = \{ \cdot , \bar{\tilde{h}}_q^{\text{disp}} \}_1,$$

where the Poisson bracket is the dispersionless limit of (2) and the dispersionless Hamiltonians are given by

$$\tilde{h}_q^{\text{disp}} = \frac{2}{(q+1)!} \operatorname{Res}_{p=0} \left[p^{-1} \mathcal{L}^{q+1} (\log \mathcal{L} - c_{q+1}) \right].$$

But these Hamiltonians satisfy a recursive relation that differs from the previous relation (4):

$$\{ \cdot , \bar{\tilde{h}}_{q-1} \}_2 = q \{ \cdot , \bar{\tilde{h}}_q \}_1 + 2 \{ \cdot , \bar{h}_{q-1} \}_1. \tag{7}$$

We briefly mention that in the dispersionless limit, all the flows can be introduced quite differently, using the relation between systems of hydrodynamic type and Frobenius manifolds [10]. The dispersionless Toda Poisson pencil is associated with a Frobenius manifold characterized by the free energy $F = u^2 v/2 + e^v$. All the Hamiltonians of the system can then be obtained by expanding the so-called deformed flat coordinates of the Frobenius manifold. The relation between the dispersive and dispersionless versions of the Toda hierarchy is just a particular instance of the classification program for bi-Hamiltonian integrable hierarchies proposed in [11] based on reconstructing the entire dispersive hierarchy starting from its dispersionless limit.

We thus see that in the dispersionless case, the Toda hierarchy has two perfectly well-defined sequences of flows, all commuting among themselves, denoted by the times t_q and $\tilde{t}_q$ for $q \geq 0$. The classical dispersive flows corresponding to the times t_q defined above reduce to the corresponding flows in the dispersionless hierarchy for $\epsilon \to 0$. In the classical dispersive formulation, on the other hand, there is apparently no flow reducing to the dispersionless flows corresponding to the times $\tilde{t}_q$ for $\epsilon \to 0$.

In analogy with the Lenard–Magri procedure for the first set of Hamiltonians, we expect to find the second set of flows of the dispersive hierarchy from another Casimir of the first Poisson bracket by applying recursive relation (7) (this time with the full dispersive brackets). The first Poisson bracket in fact admits a second Casimir: $\tilde{h}_{-1} = v$. But recursive relation (7) fails to work in this case, as can be easily verified: the reason is that $\tilde{h}_{-1}$ is a Casimir of *both* Poisson brackets. This phenomenon is called "resonance" of the Poisson pencil.

But we can introduce an ansatz for the first nontrivial Hamiltonian $\tilde{h}_0$; it was given in [12] and is such that the corresponding flow coincides with the x translation,

$$\tilde{h}_0 = u \Lambda (\Lambda - 1)^{-1} \epsilon v_x. \tag{8}$$

This Hamiltonian is nonlocal because it contains the inverse of the discrete derivative, which can be written as a formal series in ϵ,

$$(\Lambda - 1)^{-1}\epsilon v_x = \sum_{k\geq 0} \frac{B_k}{k!}(\epsilon\partial_x)^k v,$$

where the Bernoulli numbers B_k are defined by

$$\frac{x}{e^x - 1} = \sum_{k\geq 0} \frac{B_k}{k!} x^k.$$

Starting from this ansatz, we can define all the Hamiltonians $\tilde{h}_q$ using recursive relation (7) and show that they commute among themselves and with all classical Toda flows.

To have an explicit form for these new flows, it is important to find their Lax representation. Considering dispersionless Lax representation (6), we expect that introducing a logarithm of the Lax operator L is necessary. Such operator can be defined by the dressing formalism. It is well known [13] that L can be written as the dressing of the shift operators Λ and Λ^{-1},

$$L = P\Lambda P^{-1} = Q\Lambda^{-1}Q^{-1}, \tag{9}$$

where

$$P = \sum_{k\geq 0} p_k \Lambda^{-k}, \qquad p_0 = 1,$$

$$Q = \sum_{k\geq 0} q_k \Lambda^{k}.$$

By substitution in definition (9), the functions p_k and q_k can be found in terms of u and v.

Noting that $\Lambda = e^{\epsilon\partial_x}$, we are led to define two different logarithms as

$$\log_+ L = P\epsilon\partial P^{-1} = \epsilon\partial + P\epsilon P_x^{-1},$$

$$\log_- L = -Q\epsilon\partial Q^{-1} = -\epsilon\partial - Q\epsilon Q_x^{-1}.$$

These logarithms are both differential–difference operators because of the presence of $\epsilon\partial$. Seeking an expression like (6) and needing to make sense of the $(\,\cdot\,)_+$ part, we want a purely difference operator for the logarithm, which we define by

$$\log L = \frac{1}{2}\log_+ L + \frac{1}{2}\log_- L = -\frac{\epsilon}{2}(P_x P^{-1} - Q_x Q^{-1}).$$

In this definition, the derivative drops out, and we obtain a difference operator of the form

$$\log L = \sum_{k\in\mathbb{Z}} w_k \Lambda^k.$$

We want to express the coefficients w_k in terms of the variables u and v. This is indeed possible (see [1] for a proof). Essentially, all the previously defined logarithms, by definition, commute with L, e.g.,

$$[\log_+ L, L] = 0. \tag{10}$$

Substituting

$$\log_+ L = \epsilon\partial + 2\sum_{k\le -1} w_k \Lambda^k$$

in (10), we can solve it recursively and express the coefficients w_k for $k \le -1$ as formal power series in ϵ with coefficients that are differential polynomials in u and v. The analogous expression for $\log_- L$ gives the coefficients w_k for $k \ge 0$. The first few examples are

$$w_{-1} = \frac{1}{2}(\Lambda - 1)^{-1}\epsilon u_x,$$

$$w_0 = \frac{1}{2}\Lambda(\Lambda - 1)^{-1}\epsilon v_x,$$

$$w_1 = \frac{1}{2}\Lambda e^{-v}(\Lambda - 1)^{-1}\epsilon u_x.$$

We note that, in general, this is not possible for the coefficients p_k and q_k of the dressing operators.

Using this definition of the logarithm of L, we can give the Lax representation for the new flows, which is formally analogous to the dispersionless Lax representation,

$$\epsilon\frac{\partial L}{\partial \tilde{t}_q} = [A_q, L], \qquad A_q = \frac{2}{q!}[L^q(\log L - c_q)]_+,$$

Here, A_q is a difference operator of infinite order. Equivalently, we can use the operator

$$\tilde{A}_q = \frac{2}{q!}[L^q(\log L - c_q)]_+ - \frac{1}{q!}L^q(\log_- L - c_q),$$

which also contains a differential part but is of finite order. It gives the same Lax equations because it differs from A_q by a part that commutes with L. For example, the first two Lax operators are given by

$$\tilde{A}_0 = \epsilon\partial,$$

$$\tilde{A}_1 = \Lambda(\epsilon\partial - 1) + \Lambda(\Lambda - 1)^{-1}\epsilon u_x + u(\epsilon\partial - 1) + e^v(\epsilon\partial + 1 - (\Lambda - 1)^{-1}\epsilon v_x)\Lambda^{-1}.$$

As expected, the first corresponds to the x translations, and the second gives the first nontrivial extended Toda flow

$$\begin{aligned} \epsilon u_{\tilde{t}_1} &= (\Lambda - 1)(-e^v(\Lambda^{-1} - 1)^{-1}\epsilon v_x) - 2(\Lambda - 1)e^v + \epsilon\left(\frac{u^2}{2} + e^v\right)_x, \\ \epsilon v_{\tilde{t}_1} &= ((\Lambda^{-1} - 1)^{-1}\epsilon v_x)(\Lambda^{-1} - 1)u + \epsilon v_x(\Lambda^{-1}u) + \Lambda^{-1}\epsilon u_x + \epsilon u_x + 2(\Lambda^{-1} - 1)u. \end{aligned} \tag{11}$$

We can also give an explicit expression for the Hamiltonians in analogy with the dispersionless case,

$$\tilde{h}_q = \frac{2}{(q+1)!}\operatorname{Res}[L^{q+1}(\log L - c_{q+1})].$$

These are exactly the Hamiltonians defined above by the recursive relation and ansatz (8) up to total derivatives.

The new flows of this extended Toda hierarchy are nonlocal but nevertheless have many of the nice properties that are expected from a completely integrable system, such as the existence of multisolitonic solutions. For example, the evolution of a simple soliton under nonlocal flows (11) is given by

$$u(x,\tilde{t}_1) = (\Lambda - 1)\frac{\cosh((1/\epsilon)((x+\lambda_1\tilde{t}_1)\log z_1 + \tilde{t}_1(-z_1+z_1^{-1})))}{\cosh((1/\epsilon)((x+\lambda_1\tilde{t}_1)\log z_1 + \tilde{t}_1(-z_1+z_1^{-1})) - \log z_1)},$$

$$v(x,\tilde{t}_1) = (1-\Lambda^{-1})^2 \log\left[2\cosh\left(\frac{1}{\epsilon}(x+\lambda_1\tilde{t}_1)\log z_1 + \frac{1}{\epsilon}\tilde{t}_1(-z_1+z_1^{-1})\right)\right],$$

where $\lambda_1 = z_1 + z_1^{-1}$ and z_1 is a parameter. As usual, this is simply obtained by Darboux transformations of a particular constant solution.

We expect that the algebro-geometric quasiperiodic solutions should also have a nice behavior under these flows, but this problem is still under consideration. Further generalization to the multicomponent case is in progress.

All the proofs of the above statements will appear in [1].

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